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ON THE PROPAGATION OF NON-STEADY PERTURBATIONS IN A BOUNDARY LAYER WITH SELFINDUCED PRESSURE*

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The problem of supersonic flow past a flat plate bearing a triangular vibrator which begins to execute harmonic oscillations in the unperturbed boundary layer, is studied. Both the plate and vibrator are assumed to be thermally insulated. The size of the vibrator and the frequency of its oscillations are such that the flow can be described by the equations for a boundary layer with selfinduced pressure. The oscillation amplitude is assumed to be small, and this enables the equations to be linearized. The linear formulation is used to study the problem where the pressure reaches a steady-state periodic mode. The problem of the vibrator is used to solve the problem of the propagation of non-steady perturbations both upstream and downstream.

1. **Formulation of the problem and its formal solution.** Consider the flow of an ideal gas past a thermally insulated body representing a flat plate with an irregularity positioned at some distance from the ends, and of which changes its form with time. We shall assume that the parameters of the incoming unperturbed flow (U_∞^* is the velocity, p_∞^* is the pressure and ρ_∞^* is the density) determine the Mach number $M_\infty > 1$ (here and henceforth the subscript ∞ refers to the parameters in the unperturbed flow). We assume that the dependence of the first coefficient of viscosity on temperature T^* is linear $\lambda_1^*/\lambda_{1\infty}^* = CT'$ ($T' = T^*/T_\infty^*$), and the Prandtl number is equal to unity. The distance from the leading edge of the plate to the irregularity will be denoted by L^*, and in place of the reciprocal of the Reynolds number we shall use the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$ ($\mathrm{Re}_1 = \rho_\infty^* U_\infty^* L^*/\lambda_{1\infty}^*$).

Let us choose the longitudinal dimension of the irregularity $O(L^*\varepsilon^3)$, the transverse dimension $O(L^*\varepsilon^5)$, and the characteristic time of variation in the form of the irregularity $O(L^*\varepsilon^2/U_\infty^*)$. To describe the motion in the neighbourhood of such an irregularity it is convenient to separate three characteristic regions /1, 2/: the upper region of the supersonic inviscid flow ($y_1^* = O(L^*\varepsilon^3)$), the intermediate region of the conventional boundary layer ($y_2^* = O(L^*\varepsilon^4)$) and the lower region of the boundary layer with selfinduced pressure. The principal difficulties that arise in such a scheme are connected with constructing a solution in the lower region, and investigation of this region is the purpose of this paper.

Let us introduce the dependent and independent dimensionless variables used in /3, 4/ and denote by x and y the Cartesian coordinate axes, with x directed along the plate, u and v the velocity vector components along the x and y axes, p the pressure and ρ the density. By requiring that the conditions of merging with the conventional boundary layer hold as $x \to -\infty$ and $y \to \infty$, we obtain, from the Navier-Stokes equations for the pincipal terms of the expansion as $\varepsilon \to 0$, a set of equations for the unsteady boundary layer with selfinduced pressure /3–5/. We shall describe the irregularity (Fig.1) in the form

$$y_w = \sigma f(t, x), \quad \sigma \ll 1 \tag{1.1}$$

and demand that the conditions of adhesion hold on the plate and the irregularity. This gives the following relations for the approximation ($\varepsilon \to 0$) used:

$$u(t, x, y_w) = 0, \quad v(t, x, y_w) = \sigma \partial f/\partial t \tag{1.2}$$

Fig.1

*Prikl.Matem.Mekhan., 47, 5, 762-770, 1983.

The conditions that the transverse dimension of the irregularity (1.1) should be small, enable the problem to be linearized by expanding the functions sought in power series in σ

$$u = y + \sigma u_1 + \ldots, \quad v = \sigma v_1 + \ldots, \quad p = \sigma p_1 + \ldots \tag{1.3}$$

Substituting the expansion (1.3) into the Navier-Stokes equations we obtain

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad \frac{\partial p_1}{\partial y} = 0$$
$$\frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + v_1 = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} \tag{1.4}$$

and we impose the following limit and boundary conditions on its solution:

$$x \to -\infty, \quad u_1 \to 0, \quad p_1 \to 0; \quad y \to \infty, \quad u_1 \to -\int_{-\infty}^{x} p_1\, dx_1$$
$$u_1(t, x, 0) = -f(t, x), \quad v_1(t, x, 0) = \partial f(t, x)/\partial t \tag{1.5}$$

Such a formulation of the problem was used in /6/, where the author studied oscillations that were harmonic with time and which lasted for an unlimited length of time. If on the other hand we begin the study of the motion at the instant $t = 0$, then we must specify additional, time-dependent conditions

$$u_1(0, x, y) = \varphi_0(x, y) \tag{1.6}$$

The function φ_0 is assumed to be correlated with (1.4) and (1.5)

$$\varphi_0(x, 0) = -f(0, x),$$
$$\varphi_0(x, \infty) = -\int_{-\infty}^{x} p_1(0, x_1)\, dx_1, \quad \frac{\partial^2 \varphi_0(x, 0)}{\partial y^2} = \frac{\partial p_1(0, x)}{\partial x}$$

Henceforth we shall consider the flows caused by the motion of the irregularity in the unperturbed boundary layer, writing for this reason $\varphi_0(x, y) = 0$.

Equations (1.4) can be analyzed more conveniently by changing to a single equation for the function u_1

$$\frac{\partial^2 u_1}{\partial y \partial t} + y\frac{\partial^2 u_1}{\partial y \partial x} = \frac{\partial^3 u_1}{\partial y^3} \tag{1.7}$$

which, in addition to (1.5), must also satisfy the following condition in accordance with the last equation of (1.4):

$$\partial^2 u_1(t, x, 0)/\partial y^2 = \partial p_1(t, x)/\partial x \tag{1.8}$$

We will construct a solution for Eq.(1.7) with the conditions (1.5), (1.6) and (1.8), using for the functions u_1, p_1, f a Fourier transform in x and a Laplace tranform in t

$$\bar{u}_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{0}^{\infty} u_1(t, x, y)\, e^{-\omega t - ikx}\, dt\, dx, \quad \operatorname{Re}\omega > l_0 > 0, \ \operatorname{Im} k = 0$$

Then we obtain the following problem for $\bar{u}_1$ and $\bar{p}_1$:

$$\frac{\partial^3 \bar{u}_1}{\partial y^3} - (\omega + iky)\frac{\partial \bar{u}_1}{\partial y} = 0$$
$$y = 0, \ \bar{u}_1 = -\bar{f}, \quad \frac{\partial^2 \bar{u}_1}{\partial y^2} = ik\bar{p}_1; \quad y \to, \quad \bar{u}_1 \to -\frac{\bar{p}}{ik} \tag{1.9}$$

We will introduce for (1.9) the following new independent variable:

$$z = (ik)^{1/3} y + \Omega, \quad i = e^{i\pi/2}, \quad \Omega = i^{-1/3} w k^{-2/3}$$

and select for k a cut in the complex plane along the positive part of the imaginary axis, assigning the value $\arg k = 0$ to the real positive k. The third-order ordinary differential equation in problem (1.9) has three specified boundary conditions containing the parameter $\bar{p}_1$. It would seem that its solution exists for an $\bar{p}_1$, but this is not so. The condition as $y \to \infty$ is equivalent to two conditions: $\bar{u}_1$ is bounded, and equal to a specified constant. For this reason the solution of the problem can be written only for a specifed $\bar{p}_1$:

$$\bar{u}_1(\Omega, k, z) = B_1(\Omega, k)\, I(z) + B_2(\Omega, k), \quad \bar{p}_1 = -ikB_2(\Omega, k) \tag{1.10}$$
$$I(z) = \int_{z}^{\infty} \operatorname{Ai}(z_1)\, dz_1, \quad F(\Omega, k) = \operatorname{Ai}'(\Omega) + (ik)^{1/3} I(\Omega)$$
$$B_1(\Omega, k) = -(ik)^{1/3} \bar{f}(\Omega, k)/F(\Omega, k), \quad B_2(\Omega, k) = -\operatorname{Ai}'(\Omega)\, \bar{f}(\Omega, k)/F(\Omega, k)$$

($\mathrm{Ai}(\Omega)$ and $\mathrm{Ai}'(\Omega)$ denote the Airy function and its derivative). Using (1.10), we write the solution of problem (1.5)-(1.8) as

$$u_1(t,x,y)=\frac{1}{(2\pi)^{3/2}i}\int\limits_{-\infty}^{\infty}\int\limits_{l-i\infty}^{l+i\infty}\bar{u}_1 e^{ikx+\omega t}\,d\omega\,dk \tag{1.11}$$

$$p_1(t,x)=\frac{1}{(2\pi)^{3/2}i}\int\limits_{-\infty}^{\infty}\int\limits_{l-i\infty}^{l+i\infty}\bar{p}_1 e^{ikx+\omega t}\,d\omega\,dk,\quad l=\mathrm{const}>0$$

2. The problem of a vibrator: approach to the steady oscillatory mode.

The perturbations caused by the vibrator executing harmonic oscillations for an infinitely long time before the instant of time considered, were studied in /6/. We will consider the perturbations caused by a vibrator which begins to operate at the instant $t=0$ in the unperturbed boundary layer $(\varphi_0(x,y)=0)$. With this in mind, we define the function

$$f(t,x)=f_1(x)\sin\omega_0 t \tag{2.1}$$

where ω_0 is dimensionless frequency and the function $f_1(x)$ describes, as in /7/, a triangular form (Fig.1) with parameters a and b ($f_1(x)=0$ at $x\leqslant 0$, $2x$ at $0\leqslant x\leqslant b$, $2b(a-x)/(a-b)$ at $b\leqslant x\leqslant a$, 0 at $x\geqslant a$). Let us investigate the pressure in the gas caused by such a vibrator. According to (1.10),(1.11) and (2.1) we have

$$p_1=-\frac{\omega_0}{2\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{l-i\infty}^{l+i\infty}\frac{\bar{f}_0(a,b,k)\,\mathrm{Ai}'(\Omega)}{k(\omega_0^2+\omega^2)F(\Omega,k)}e^{\omega t+ikx}\,d\omega\,dk \tag{2.2}$$

$$\bar{f}_0(a,b,k)=1-\frac{a}{a-b}\exp(-ikb)+\frac{b}{a-b}\exp(-ika)$$

We will use (2.2) as the starting point to obtain an asymptotic formula for the pressure p_1 at finite x and $t\to\infty$. To do this we consider the inner integral

$$J_t=\int\limits_{l-i\infty}^{l+i\infty}\frac{\mathrm{Ai}'(\Omega)e^{\omega t}\,d\omega}{(\omega_0^2+\omega^2)F(\Omega,k)}$$

The denominator of the integrand vanishes when $\omega=i\omega_0,\ \omega=-i\omega_0$, and we then have

$$F(\Omega,k)=\mathrm{Ai}'(\Omega)+(ik)^{1/3}I(\Omega)=0 \tag{2.3}$$

Equation (2.3) represents a dispersion relation connecting ω with k for free oscillations in a supersonic boundary layer /4, 5/.

A computer was used to construct the trajectories of the roots $\omega(k)$ when k varies along the real axis. When $k<0$, the trajectories $\omega(k)$ lie in the second quadrant (Fig.2) and the inequality $\pi>\arg\omega(k)>0.57\pi$ holds for all roots; when $k>0$, the trajectories $\omega(k)$ are symmetrical about the axis $\mathrm{Im}\,k=0$ with respect to the trajectories shown in Fig.2, lie in the third quadrant and satisfy the inequality $-0.57\pi>\arg\omega(k)>-\pi$. the Cauchy theorem

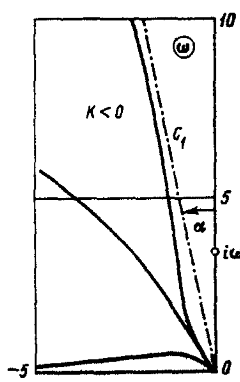


Fig.2

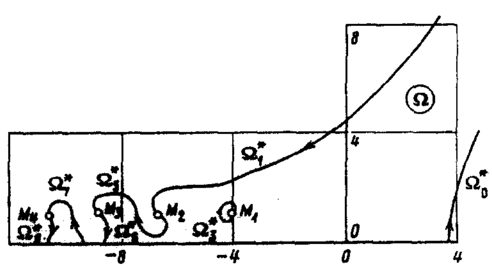


Fig.3

on residues, we change the path of integration in J_t, putting $l=0$ and choosing two rays as the new integration path, namely C_1 shown in Fig.2 and, symmetrical to it, C_2 lying in the third quadrant. Let the angles made by the rays C_1 and C_2 with the positive direction of the axis $\mathrm{Im}\,\omega=0$ be $\pi/2+\alpha$ and $-\pi/2-\alpha$, where α satisfies the inequality $0<\alpha<0.07$. When the integration path is changed in this manner, the poles of the points $i\omega_0$ and $-i\omega_0$, must be taken into account, but not the poles corresponding to the roots (2.3). As a result we have

$$J_t = 2\pi i \left[\frac{e^{i\omega_0 t}\,\mathrm{Ai}'(\Omega_1)}{2i\omega_0 F(\Omega_1, k)} - \frac{e^{-i\omega_0 t}\,\mathrm{Ai}'(\Omega_2)}{2i\omega_0 F(\Omega_2, k)} \right] + \left(\int\limits_{\infty e^{-i\varphi}}^{0} + \int\limits_{0}^{\infty e^{i\varphi}} \right) \frac{\mathrm{Ai}'(\Omega)\, e^{\omega t}\, d\omega}{(\omega_0^2 + \omega^2)\, F(\Omega, k)} \tag{2.4}$$

$$\Omega_1 = \frac{i^{1/3}\omega_0}{k^{2/3}}, \quad \Omega_2 = -\Omega_1, \quad \varphi = \pi/2 + \alpha$$

Let us assess the contribution of the integrals on the right-hand side of (2.4) to the pressure p_1, as $t \to \infty$. The integrand in these integrals is written in a form suitable for application of Laplace's lemma on the asymptotic estimation of integrals according to which the principal contribution in the course of integration will be made by the small neighbourhood of the point 0. Using further the fact that $|I(\Omega)/\mathrm{Ai}'(\Omega)|$ is bounded when ω varies along the integration path we find, that the contribution towards p_1 as $t \to \infty$ will be of the order of $o(t^{-3})$.

Writing the integrals in k we obtain, from the first term determined by the poles $i\omega_0$ and $-i\omega_0$,

$$P_1 = -\frac{1}{\pi}\cos\omega_0 t \int\limits_{-\infty}^{\infty} \mathrm{Re}(\Phi_0)\, dk + \frac{1}{\pi}\sin\omega_0 t \int\limits_{-\infty}^{\infty} \mathrm{Im}(\Phi_0)\, dk + o(t^{-3}) \tag{2.5}$$

$$\Phi_0 = \bar{f}_0(a, b, k)\,\mathrm{Ai}'(\Omega_1) k^{-1} F^{-1}(\Omega_1, k) e^{ikx}$$

Remembering that the motion studied in /7/ was generated by a vibrator oscillating according to the law $\cos\omega_0 t_7$ (the time used in /7/ is denoted by t_7), while the motion studied here obeys the law $\sin\omega_0 t$ and hence $t_7 = t - \pi/2\omega_0$, we find that the formula for the pressure in /7/ agrees with (2.5) to within quantities $o(t^{-3})$. This shows that, after the vibrator commences working, the steady state oscillatory mode in the gas is reached fairly quickly, more quickly than according to the law t^{-3}.

3. **On the propagation of non-steady perturbations.** Using the problem of the vibrator, we shall consider the question of the propagation of non-steady perturbations in a supersonic boundary layer with selfinduced pressure. With this aim we again turn to Eq.(2.2), which defines the pressure, and study the asymptotic behaviour of p_1 at finite t and $|x| \to \infty$. We change the order of integration in (2.2), choosing the integral in k as the inner one. This can be done using Fubini's theorem, since the integrand decreases rapidly as $|k| \to \infty$ and $|\omega| \to \infty$. We will use a computer to study the behaviour of the roots $k(\omega)$ in (2.3) with ω varying along the straight line parallel to the imaginary axis with $l \gg 1$ ($\omega = l + il_1$, $-\infty < l_1 < \infty$, $l = \mathrm{const}$). Since the Riemannian surface Ω consists of a single sheet, we will first consider the roots $\Omega(\omega)$. As was stated in /6/, we have a denumerable set of roots for a definite value of ω (e.g. $l = 0, l_1 = 1$), consequently each root generates a trajectory when ω changes. Using the notation for the roots adopted in /6/, we denote the corresponding trajectories by Ω_1^*, Ω_j^*, where $j = 0, 1, \ldots$. The pattern of the first trajectories characteristic for large values of l is shown in Fig.3 for $l = 7.5$ only for $\mathrm{Im}\,\Omega > 0$, since for $\mathrm{Im}\,\Omega < 0$ the trajectories are symmetrical, with respect to those given in Fig.3, about the axis $\mathrm{Im}\,\Omega = 0$. The arrows indicate the directions corresponding to increasing l_1. M_j denote the zeros of $I(\Omega)$, while the corresponding, symmetrically distributed points in the third quadrant which are also the roots of $I(\Omega)$, will be denoted by M_j^-. Equation (2.3) implies that the trajectories of its roots tend to the points M_j and M_j' as $l_1 \to \pm\infty$. Amongst the trajectories we have the trajectory Ω_0^* emerging from the point a infinity, and returning to it. Two trajectories, Ω_1^* and Ω_1^{**} connect the point at infinity with the points M_j and M_j', and j increases as l increases. Thus, when $l < 6.8$, the trajectories connect the point at infinity with M_1 and M_1'. When the trajectories Ω_1^* and Ω_1^{**} pass from the points M_j and M_j' to M_{j+1}, M_{j+1}', a single trajectory emerges from the points $M_j, M_j', \ldots, M_1$ and M_1' and returns to them, e.g. Ω_3^* for M_1 and Ω_2^* for M_1'. As regards the points M_{j+2}, M_{j+2}', M_{j+3} and $M_{j+3}', \ldots$, they are connected to one another by the trajectories, with the trajectories Ω_7^* and Ω_8^* connecting the points M_4 and M_4'.

Let us now consider the perturbations propagating upstream as $x \to -\infty$. With this aim we inspect the inner integral in k

$$J_x = \int\limits_{-\infty}^{\infty} \Phi_x\, dk = \int\limits_{-\infty}^{\infty} \frac{\bar{f}_0(a, b, k)\,\mathrm{Ai}'(\Omega)\, e^{ikx}}{kF(\Omega, k)}\, dk \tag{3.1}$$

Using the relation $k = i^{-1}\omega^{-3/2}\Omega^{-3/2}$ and the trajectories $\Omega(\omega)$ obtained and shown in Fig.3, we construct the roots $k(\omega)$. For the lower half-plane $(-\pi < \arg k < 0)$ we have a single trajectory $k_0^*(\omega)$ (Fig.4) generated by the trajectory $\Omega_0^*(\omega)$. We compute the integral (3.1) using a closed contour consisting of a segment of the real axis from $-r$ to r, and the arc of the circle lying in the lower half-plane and connecting the points $-r$ and r (Fig.4)

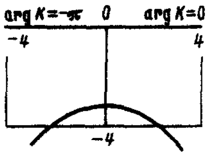


Fig.4

Since the integral along the arc of the circle tends to zero as $r \to \infty$ when $x < 0$, then according to Cauchy's theorem the integral (3.1) is given by the residue at the point $k_0^*(\omega)$. Substituting the value of J_x obtained into (2.2), we obtain

$$p_1 = -\frac{\omega_0}{2\pi^2}\int_{l-i\infty}^{l+i\infty} \frac{\bar{f}_0(a, b, k_0^*(\omega))}{\omega_0^2+\omega^2} B(\omega, k_0^*(\omega))\, e^{\omega t + ik_0^*(\omega)x}\, d\omega \tag{3.2}$$

$$B(\omega, k_0^*(\omega)) = \frac{3\pi \mathrm{Ai}'(\Omega_0^*)\, t^{1/3} k_0^{*1/3}}{2I(\Omega_0^*) + \Omega_0^*\left(1 + \frac{\omega}{k_0^{*2}}\right)\mathrm{Ai}(\Omega_0^*)}$$

We use the method of steepest descent as $x \to \infty$, express the integral (3.2) as a sum of three integrals corresponding to three terms in the function $\bar{f}_0$, and find the roots of the equation

$$\frac{d}{d\varphi}(\omega t - ik_0^*(\omega)|x_1|) = 0 \quad \text{or} \quad \frac{dk_0^*}{d\omega} = -\frac{it}{|x_1|} \tag{3.3}$$

where $x_1 = x;\ x - b;\ x - a$ for the first, second and third integral respectively. According to /5/ the quantity $|dk_0^*/d\omega| \to 0$ as $|\omega| \to \infty$, and the following expression holds for k_0^*:

$$k_0^* = -i\omega^{1/2} - {}^1/_2 i\omega^{-1/2} + O(\omega^{-3/2}) \tag{3.4}$$

Substituting (3.4) into (3.3) we find the root of (3.3), i.e. the point of steepest descent

$$\omega = \omega' = \frac{x_1^2}{4t^2} - 1 + O\left(\left(\frac{t}{x_1}\right)^2\right)$$

and draw through it the contour of integration, choosing for this $l = \omega'$. Since the relation $|\omega| \gg 1$ always holds on such a path of integration, it follows that the root k_0^* can be replaced by the series (3.4). Making the change of variable $\omega = \omega_1 x_1^2/t^2$, we write the function appearing in the exponent in (3.2) in the form

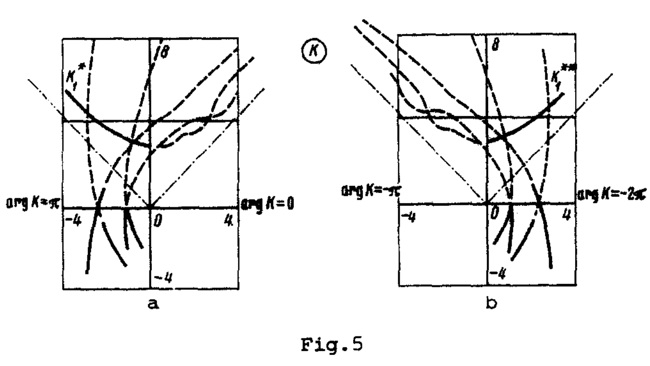
$$g_0 = x_1^2 t^{-1} g_1(t, x_1, \omega_1) + O(t^3 x_1^{-2})$$
$$g_1 = \omega_1 - \omega_1^{1/2} - {}^1/_2 t^2 x_1^{-2}\omega_1^{-1/2}$$

The real part of the function g_1 takes a maximum value on the integration path at the point $\omega_1 = \omega_1' = {}^1/_4 - t^2/x_1^2$, and the integration path coincides, in the neighbourhood of this point, with the line $\mathrm{Im}\, g_1(\omega_1) = \mathrm{Im}\, g_1(\omega_1')$. Then the integral (3.2) will be determined, as $x_1^2/t \to \infty$, by the neighbourhood of the point $\omega_1 = \omega_1'$ and we can write, in accordance with the method of steepest descent,

$$p_1 = 8\pi^{-1/2}\omega_0 t^{5/2} e^{-t}\left[\Phi_1(x) - \frac{a}{a-b}\Phi_1(x-b) + \frac{b}{a-b}\Phi_1(x-a)\right] \sim 8\pi^{-1/2}\omega_0 t^{5/2}|x|^{-3}\exp(-x^2/(4t) - t) \tag{3.5}$$

$$\Phi_1(x) = |x|^{-3}\exp(-x^2/4t)\,(1 + O((t^2 + t^3)/x^2))$$

From (3.5) it follows that the perturbations propagate upstream with infinite speed and appear, at any $t > 0$, at once over the whole region $x < 0$ with the fronts completely absent from the flow (such as a moving front of the discontinuities in the first-order derivatives). The perturbations are unaffected by the geometrical form of the vibrator,



a b

Fig.5

since a and b do not occur in the principal term, which is proportional to the initial normal velocity ω_0 of the vibrator surface. Such a mode of propagation is typical for the perturbations described by parabolic equations.

Let us now consider the perturbations propagating downstream as $x \to \infty$. Again we consider the integral J_x given by (3.1). Using the trajectories $\Omega(\omega)$ (Fig.3) we construct the roots $k(\omega)$ in the upper half-plane for $0 \leqslant \arg k \leqslant \pi$ in Fig.5a and for $-2\pi \leqslant \arg k \leqslant -\pi$ in Fig.5b. The solid lines in Fig.5a depict the roots for which $0 \leqslant \arg \omega \leqslant \pi/2$ holds, and the dashed lines the roots for which $-\pi/2 \leqslant \arg \omega \leqslant 0$ holds. The solid lines in Fig.5b represent the trajectories for which $-\pi/2 \leqslant \arg \omega \leqslant 0$ holds, and the dashed lines those for which $0 \leqslant \arg \omega \leqslant \pi/2$ holds. Taking the behaviour of the roots into account, we transform the integral J_x, first for ω satisfying the inequality $0 \leqslant \arg \omega \leqslant \pi/2$. We split J_x into two integrals

$$J_x = J_{x1} + J_{x2}, \quad J_{x1} = \int_{-\infty}^{0} \Phi_x\, dk, \quad J_{x2} = \int_{0}^{\infty} \Phi_x\, dk$$

In accordance with Fig.5b we replace the integration path in J_{x1} by a ray emerging at the angle $-\pi - \alpha_1$ without grazing the trajectories of the roots $k(\omega)$ (the dashed line). On the other hand, according to Fig.5a, in the interval J_{x2}, we replace the path of integration by a ray emerging at an angle $\pi - \alpha_1$ without grazing the trajectory of the roots $k(\omega)$ and also take into account the residue generated by the root $k_1^*(\omega)$. Asymptotic analysis at $l \gg 1$ shows that such α_1 do exist. As a result we have

$$J_x = \int_0^\infty \exp(-ie^{-i\alpha_1}qx)\, \bar{J}_0(a, b, -e^{-i\alpha_1}q)\left[\frac{-\mathrm{Ai}'(\Omega_3)}{F(\Omega_3, e^{-i\pi - i\alpha_1}q)} + \right. \tag{3.6}$$

$$\left.\frac{\mathrm{Ai}'(\Omega_4)}{F(\Omega_4, e^{i\pi - i\alpha_1}q)}\right]\frac{dq}{q} + 2\pi \operatorname{res} \Phi_x(k_1^*(\omega))$$

$$\Omega_3 = \omega q^{-2/3} \exp(\pi i/3 + 2i\alpha_1/3), \quad \Omega_4 = \Omega_3 \exp(-4i\pi/3)$$

Applying to the right-hand side of (3.6) transformations analogous to those used in /7/ in computing the asymptotic forms as $x \to \infty$, we find that when $x \to \infty$,

$$J_x = \Phi_2(\omega, x, a, b) + 2\pi i \operatorname{res} \Phi_x'(k_1^*(\omega)) \tag{3.7}$$

$$\Phi_2(\omega, x, a, b) = -i2^{2/3}3^{-1/2}\pi^{1/2}\omega^{1/2}\left[\Phi_3(\omega, x) - \frac{a}{a-b}\Phi_3(\omega, x-b) + \frac{b}{a-b} I_3(\omega, x-a)\right]$$

$$\Phi_3(\omega, x) = x^{-1}\exp(-2^{2/3}3^{-1/2}\omega^{3/4}x^{3/2})\left[1 + O(x^{-1/2}\omega^{-3/4})\right] \times \left[1 - \frac{2}{3}x^{-1}\omega^{1/2} - \sqrt{\frac{2}{3}}\, x^{-1/2}\omega^{-3/4}\left(\frac{7}{48} - \frac{41}{72}x^{-1}\omega^{3/2}\right)\right]^{-2}$$

Next we transform J_x in the range $-\pi/2 \leqslant \arg \omega \leqslant 0$ as follows. We choose for the integration path in J_{x1} the ray emerging at the angle $2\pi + \alpha_1$, and take into account the residue generated by the root $k_1^{**}(\omega)$, while in the integral J_{x2} we choose, as the integration path, the ray emerging at an angle α_1. New integration paths are shown in Fig.5a and b by the dot-dash lines. Carrying out transformations analogous to those for $0 \leqslant \arg \omega \leqslant \pi/2$, we find, that as $x \to \infty$ and $-\pi/2 \leqslant \arg \omega \leqslant 0$ the integral

$$J_x = \Phi_2(\omega, x, a, b) + 2\pi i \operatorname{res} \Phi_x(k_1^{**}(\omega)) \tag{3.8}$$

Substituting expressions (3.7) and (3.8) into (2.2), we obtain

$$p_1 = -\frac{\omega_0}{2\pi^2}\left[I_1(t, x, a, b) + 2\pi i I_2(t, x, a, b)\right] \tag{3.9}$$

$$I_1 = \int_{l-i\infty}^{l+i\infty} \Phi_2(\omega, x, a, b)\frac{e^{\omega t}\, d\omega}{\omega_0^2 + \omega^2}$$

$$I_2 = \int_{l-i\infty}^{l} \operatorname{res} \Phi_x(k_1^{**}(\omega))\frac{e^{\omega t}\, d\omega}{\omega_0^2 + \omega^2} + \int_{l}^{l+i\infty} \operatorname{res} \Phi_x(k_1^{*}(\omega))\frac{e^{\omega t}\, d\omega}{\omega_0^2 + \omega^2}$$

In analyzing the integral I_1 we shall only consider the principal term proportional to $\Phi_3(\omega, x-a)$. Making the change of variable $\omega = \omega_2 (x-a)^2 t^{-4}$, we obtain

$$I_1 \sim -i2^{2/3}3^{-1/2}\pi^{1/2} b(a-b)^{-1}(x-a)^2 t^{-6} \int_{l'-i\infty}^{l'+i\infty} \omega_2^{1/2} \times$$

$$\left[1 + O((x-a)^{-2}t^3\omega_2^{-3/4})\right](\omega_0^2 + \omega_2^2(x-a)^4 t^{-8})^{-1} \times$$

$$\left[1 - \frac{2}{3}t^{-2}\omega_2^{1/2} - \sqrt{\frac{2}{3}}(x-a)^{-2}t^3\omega_2^{-3/4}\left(\frac{7}{48} + \frac{41}{72}t^{-2}\omega_2^{3/2}\right)\right]^{-2} \times$$

$$\exp\left[(x-a)^2 t^{-3}(\omega_2 - 2^{2/3}3^{-1/2}\omega_2^{3/4})\right] d\omega_2, \quad l' = lt^4(x-a)^{-2}$$

Passing to the limit as $(x-a)^2/t^3 \to \infty$, we will estimate the integral I_1 using the method of steepest descent. The point $\omega_2 = {}^9/_4$, is the point of steepest descent, therefore we choose $l' = {}^9/_4$. On such an integration path the real part of the function under the exponent sign will attain its maximum value at the point of steepest descent, and the imaginary part of this function will remain constant in the neighbourhood of this point. This gives

$$I_1 \sim 2^5 3^{-5/2} \pi b (a-b)^{-1} (x-a)^{-3} t^{7/2} \Big[1 - t^{-2} - \frac{4}{9}(x-a)^{-2} \times t^3 \Big(\frac{7}{48} + \frac{41}{48} t^{-2}\Big)\Big]^{-2} [1 + O(t^3(x-a)^{-2})] \exp[-3(x-a)^2/4t^3] \quad (3.10)$$

Let us expand the integrands in I_2 for large ω, again restricting ourselves to the principal term proportional to $b/(a-b)$

$$I_2 = \frac{b}{2(a-b)} \left\{ \int\limits_{l}^{l+i\infty} \frac{(k_1^*)^2 \omega}{\omega_0^2 + \omega^2} e^{\omega t + ik_1^*(x-a)} [1 + O(\omega^{-1})] d\omega + \int\limits_{l-i\infty}^{l} \frac{(k_1^{**}) \omega}{\omega_0^2 + \omega^2} e^{\omega t + ik_1^{**}(x-a)} [1 + O(\omega^{-1})] d\omega \right\} \quad (3.11)$$

The following expression holds for :

$$k_1^* = \omega^{1/2} e^{\pi i/2} (1 - {}^1/_2 \omega^{-1} + O(\omega^{-2}))$$
$$k_1^{**} = \omega^{1/2} e^{-3\pi i/2} (1 - {}^1/_2 \omega^{-1} + O(\omega^{-2}))$$

Since the principal parts of the integrals in (3.11) contain only integral powers of k_1^* and k_1^{**}, it follows that we can write a single integral of an analytic function with the limits $l - i\infty$ and $l + i\infty$. For the real part of the function under the exponential sign $\omega t - \omega^{1/2}(1 - {}^1/_2 \omega^{-1} + \ldots)(x-a)$, the point of steepest descent will be

$$\omega = (x-a)^2/(4t^2) + 1 + O(t^2(x-a)^{-2})$$

Carrying out computations analogous to those used in determining the asymptotic forms for p_1 as $x \to -\infty$, we obtain the asymptotic form of the expression for I_2 as $(x-a)^2 t^{-1} \to \infty$. Substituting the asymptotic expressions for I_1 (3.10) and I_2 (3.9), we obtain the asymptotic form for p_1 as $x \to \infty$:

$$p_1 = -2^4 3^{-5/2} \pi^{-1} \omega_0 b (a-b)^{-1} (x-a)^{-3} t^{7/2} [1 + O(t^3(x-a)^{-2}] \times \Big[1 - t^{-2} - \frac{4}{9}(x-a)^{-2} t^3 \Big(\frac{7}{48} + \frac{41}{48} t^{-2}\Big)\Big]^{-2} \times \exp[-3(x-a)^2/4t^3] + 8\pi^{-1/2} \omega_0 b (a-b)^{-1} (x-a)^{-3} t^{5/2} \times [1 + O((t^2 + t^3)/x^2] \exp[-(x-a)^2/4t + t] \quad (3.12)$$

It follows from (3.12) that the perturbations propagate downstream with infinite speed and there are no fronts in the flow. The perturbations at $x \to \infty$ become more complicated than those at $x \to -\infty$. At the time $t < \sqrt{3}$ the character of the perturbations is determined by the second term of (3.12) which defines the perturbations analogous to those appearing when $x \to -\infty$ (3.5), while for $t > \sqrt{3}$ the first term of (3.12) is the principal term, i.e. the asymptotic modes change as $x \to \infty$.

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